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2-D Dynamic Cavity Expansion Model for Arbitrary Tractions

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A 2-D dynamic cavity expansion model is developed for the cylindrical cavity in an infinite medium subjected to an asymmetric constant velocity vector as a surface traction on the cavity wall. The equation of motion for an elemental volume of elastic material in plane strain and in polar coordinates is used. The analysis dealing with the exterior of the cavity consists of Hankel functions of the first kind. The Least square procedures and the Fourier transforms are employed for the elastic problems. As a first technique, Least square procedures has been used to solve the unknown constants of the elastic solutions with the initial conditions on the cavity surface. Algorithms of inverse fast Fourier transforms (FFT) will be an efficient technique for this case. For each discrete time step, the equilibrium equations, along with the von-Mises yield condition, are solved for the solutions in the plastic region. The model can be also applied when arbitrary normal and tangential tractions acting on the surface of the circular cavity are prescribed.

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2-D Dynamic Cavity Expansion Model for Arbitrary Tractions

Hyung Je Woo

Abstract

A two dimensional dynamic cavity expansion model is developed for the cylindrical cavity in an infinite medium subjected to an asymmetric constant velocity vector as a surface traction on the cavity wall. From the theory of elasticity, the equation of motion for an elemental volume of elastic material in plane strain and in polar coordinates is used. The analysis dealing with the exterior of the cavity consists of Hankel functions of the first kind. Two different methods, the Least square procedures and the Fourier transforms, are employed for the elastic problems. As a first technique, Least square procedures has been used to solve the unknown constants of the elastic solutions with the initial conditions on the cavity surface. The frequency equations can be solved, for example, for the first ten roots for each of the first eleven modes of vibration. Secondly, Fourier transforms are used to express all the solutions in the frequency domain and then the complete solutions can be obtained by inversing these Fourier transforms. Algorithms of inverse fast Fourier transforms (FFT) will be an efficient technique for this case. For each discrete time step, the equilibrium equations, along with the von-Mises yield condition, are solved for the solutions in the plastic region. The model can be also applied when arbitrary normal and tangential tractions acting on the surface of the circular cavity are prescribed.

1.0 Elastic Region

1.1 Wave Equations

The equations of motion of an isotropic elastic solid in which body forces are absent, are:

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda+\mu)\frac{\partial\Delta}{\partial x} + \mu\nabla^2 u \tag{1a}$$

$$\rho\frac{\partial^2 v}{\partial t^2} = (\lambda+\mu)\frac{\partial\Delta}{\partial y} + \mu\nabla^2 v \tag{1b}$$

$$\rho\frac{\partial^2 w}{\partial t^2} = (\lambda+\mu)\frac{\partial\Delta}{\partial z} + \mu\nabla^2 w, \tag{1c}$$

where the operator ∇^2 is written for $(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})$.

Using the vector operators with the displacement $\boldsymbol{u}$, we have another form of the vector displacement equation of motion as shown below:

$$(\lambda + 2\mu)\nabla(\nabla \bullet \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u} = \rho\ddot{\mathbf{u}} \tag{2}$$

where the dilatation is $\Delta = \nabla \cdot \mathbf{u}$.

These equations of motion may be shown to correspond to the propagation of two types of waves through the medium

$$\rho\frac{\partial^2 \Delta}{\partial t^2} = (\lambda + 2\mu)\nabla^2 \Delta \tag{3a}$$

$$\rho\frac{\partial^2 \mathbf{w}}{\partial t^2} = \mu\nabla^2 \mathbf{w}, \tag{3b}$$

where Δ is the dilatation propagating through the medium with the velocity $[(\lambda + 2\mu)/\rho]^{1/2}$ and $\mathbf{w}$ is the rotation about the axis propagating with the velocity $(\mu/\rho)^{1/2}$. In vector forms,

$$\nabla^2 \Delta = \ddot{\Delta} / c_d^{\,2} \tag{4a}$$

$$\nabla^2 \mathbf{w} = \ddot{\mathbf{w}} / c_s^{\,2}. \tag{4b}$$

Hence the displacement equation of motion governs waves of the dilatation and the rotation.

Poisson gave a class of particular solutions to Eqn. (2), based on the assumption that the displacement vector $\mathbf{u}$ is the gradient of a potential function, i.e., $u = \nabla\phi$. Later, Lame gave the general solution using both a scalar and vector potential, such as every displacement vector field of the form

$$\mathbf{u} = \nabla\phi + \nabla \times \vec{\psi}, \tag{5}$$

which satisfies Eqn. (2), provided ϕ and $\vec{\psi}$ are solutions of:

$$\nabla^2 \phi = \ddot{\phi} / c_d^{\,2} \tag{6a}$$

$$\nabla^2 \vec{\psi} = \ddot{\vec{\psi}} / c_s^{\,2}. \tag{6b}$$

When solutions are found to Eqn. (6) for ϕ and $\vec{\psi}$, displacement vector $\mathbf{u}$ can be obtained from Eqn. (5), and this is a solution to the displacement equations of motion.

We particularly consider the 2-dimensional plane strain problem and use the cylindrical coordinates (r, θ, z). In the absence of the body force, two scalar wave equations are:

$$\nabla^2 \phi = \ddot{\phi} / c_d^{\,2} \tag{7a}$$

$$\nabla^2 \psi = \ddot{\psi} / c_s^{\,2}. \tag{7b}$$

In cylindrical coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

For harmonic waves of frequency ω, we write

$$\phi(r,\theta,t) = \Phi(r,\theta)e^{-i\omega t} \tag{8a}$$

$$\psi(r,\theta,t) = \Psi(r,\theta)e^{-i\omega t}. \tag{8b}$$

Since the potentials must be periodic in θ with period 2π, we seek solutions of the form:

$$\Phi(r,\theta) = R(r)e^{in\theta} \tag{9a}$$

$$\Psi(r,\theta) = S(r)e^{in\theta} \tag{9b}$$

where n = integer.

Substituting Eqn. (8) into Eqn. (7), we obtain two Bessel equations:

$$R'' + \frac{1}{r}R' + (\alpha^2 - \frac{n^2}{r^2})R = 0 \tag{10a}$$

$$S'' + \frac{1}{r}S' + (\beta^2 - \frac{n^2}{r^2})S = 0 \tag{10b}$$

where $\alpha = \frac{\omega}{c_d}$ and $\beta = \frac{\omega}{c_s}$.

The general solutions of these equations are the linear combinations of Bessel functions J_n and Y_n, e.g.,

$$R(r) = AJ_n(\alpha r) + BY_n(\alpha r) \tag{11}$$

where the Bessel functions of the first and second kind and order n are given by (when p = n = 0 or integer):

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+p}}{m!(m+p)!} \tag{12a}$$

$$J_{-p}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m-p}}{m!(m-p)!} \tag{12b}$$

$$Y_p(x) = \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}. \tag{12c}$$

For brevity we employ the notation:

$$C_n^{(1)}(z) = J_n(z)$$

$$C_n^{(2)}(z) = Y_n(z)$$

$$C_n^{(3)}(z) = H_n^{(1)}(z) = J_n(z) + iY_n(z) \tag{13}$$

$$C_n^{(4)}(z) = H_n^{(2)}(z) = J_n(z) - iY_n(z)$$

where $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ are the Hankel functions of the first and second kind, respectively. The functions J_n and Y_n are suitable for the solutions of the interior boundary value problems, while $H_n^{(1)}$ and $H_n^{(2)}$ are suitable for the exterior boundary value problems.

When a cylindrical cavity is subjected to dynamic surface tractions along the axis of the cylinder, the state of strain is planar. The general solution of an arbitrary cross sectional cavity problem in an infinite medium will have the Hankel function $H_n^{(1)}$ for the cylinder function that is to be retained in the expressions of the Lame potentials.

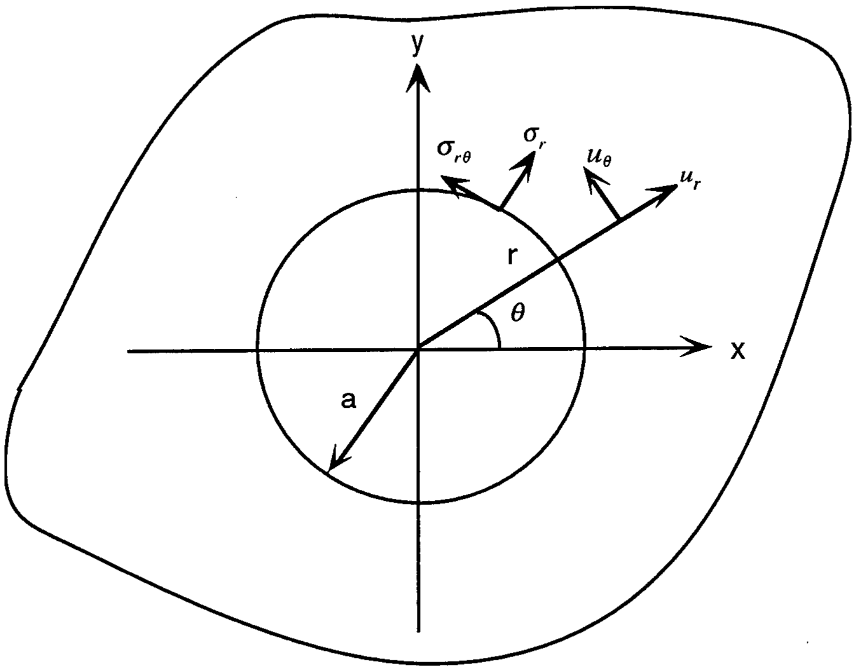

Figure 1. Cylindrical cavity with surface tractions.

The use of $H_n^{(1)}$ ensures that the Sommerfeld radiation condition is satisfied at infinity, so that $H_n^{(1)}(r)\exp(-i\omega t)$ represents waves advancing toward infinity from the surface of the cavity. Note that for waves converging to the cavity, one must employ $H_n^{(2)}(r)\exp(-i\omega t)$. Thus for the problem of a cavity subjected to surface loads we use $C_n^{(3)} = H_n^{(1)}$.

$$\Phi(r,\theta) = \sum_{n=0}^{\infty} A_n H_n^{(1)}(\alpha r) e^{in\theta} \tag{14a}$$

$$\Psi(r,\theta) = \sum_{n=0}^{\infty} B_n H_n^{(1)}(\beta r) e^{in\theta} \tag{14b}$$

Generally, we have

$$\nabla \phi = \frac{\partial \phi}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \phi}{\partial z}\mathbf{e_z}, \tag{15}$$

$$\nabla \times \psi = (\frac{1}{r}\frac{\partial \psi_z}{\partial \theta} - \frac{\partial \psi_\theta}{\partial z})\mathbf{e_r} + (\frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r})\mathbf{e}_\theta + \{\frac{1}{r}\frac{\partial}{\partial r}(r\psi_\theta) - \frac{1}{r}\frac{\partial \psi_r}{\partial \theta}\}\mathbf{e_z}, \tag{16}$$

and since $u_z = 0$ and $\frac{\partial}{\partial z} = 0$,

$$u_r = \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \psi_z}{\partial \theta} \tag{17a}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} - \frac{\partial \psi_z}{\partial r}. \tag{17b}$$

The displacement field, aside from a factor exp($-i\omega t$), becomes (see Appendix A):

$$u_r = \frac{1}{r}\sum_{n=0}^{N}\sum_{k=1}^{L}[A_{nk}U_1^{(3)}(\alpha_{nk}r) + iB_{nk}U_2^{(3)}(\beta_{nk}r)]e^{in\theta} \tag{18a}$$

$$u_\theta = \frac{1}{r}\sum_{n=0}^{N}\sum_{k=1}^{L}[iA_{nk}V_1^{(3)}(\alpha_{nk}r) + B_{nk}V_2^{(3)}(\beta_{nk}r)]e^{in\theta} \tag{18b}$$

where

$$\begin{aligned} U_1^{(i)}(\alpha r) &= \alpha r C_{n-1}^{(i)}(\alpha r) - nC_n^{(i)}(\alpha r) \\ U_2^{(i)}(\beta r) &= \mp nC_n^{(i)}(\beta r) \\ V_1^{(i)}(\alpha r) &= \mp nC_n^{(i)}(\alpha r) \\ V_2^{(i)}(\beta r) &= -\beta r C_{n-1}^{(i)}(\beta r) + nC_n^{(i)}(\beta r). \end{aligned} \tag{19}$$

In deriving Eqn. (19) we have employed the recurrence relations for the cylinder functions $C_n(z)$, where z is complex and n is any number (not necessarily an integer):

$$zC_n'(z) = nC_n(z) - zC_{n+1}(z) = -nC_n(z) + zC_{n-1}(z). \tag{20}$$

The physical components of the stress tensor follow from Hooke's law, namely,

$$\begin{aligned}
\sigma_r &= \lambda\nabla^2\phi + 2\mu\left(\frac{\partial^2\phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta}\right) \\
\sigma_\theta &= \lambda\nabla^2\phi + \frac{2\mu}{r}\left(\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\psi}{\partial\theta} - \frac{\partial^2\psi}{\partial r\partial\theta}\right) \\
\sigma_z &= \lambda\nabla^2\phi \\
\sigma_{r\theta} &= \mu\left(\frac{2}{r}\frac{\partial^2\phi}{\partial r\partial\theta} - \frac{2}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right) \\
\sigma_{rz} &= 0 \\
\sigma_{\theta z} &= 0
\end{aligned} \tag{21}$$

where $\mu = \frac{E}{2(1+\nu)}$ and $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{2\nu}{1-2\nu}\mu$.

Thus, aside from the exponential factor exp($-i\omega t$), we have (Appendix B):

$$\begin{aligned}
\sigma_r &= \frac{2\mu}{r^2}\sum_{n=0}^{N}\sum_{k=1}^{L}[A_{nk}T_{11}^{(3)}(\alpha_{nk}r) + iB_{nk}T_{12}^{(3)}(\beta_{nk}r)]e^{in\theta} \\
\sigma_\theta &= \frac{2\mu}{r^2}\sum_{n=0}^{N}\sum_{k=1}^{L}[A_{nk}T_{21}^{(3)}(\alpha_{nk}r) + iB_{nk}T_{22}^{(3)}(\beta_{nk}r)]e^{in\theta} \\
\sigma_z &= \frac{2\mu}{r^2}\sum_{n=0}^{N}\sum_{k=1}^{L}A_{nk}T_{51}^{(3)}(\alpha_{nk}r)e^{in\theta} \\
\sigma_{r\theta} &= \frac{2\mu}{r^2}\sum_{n=0}^{N}\sum_{k=1}^{L}[iA_{nk}T_{41}^{(3)}(\alpha_{nk}r) + B_{nk}T_{42}^{(3)}(\beta_{nk}r)]e^{in\theta} \\
\sigma_{rz} &= 0 \\
\sigma_{\theta z} &= 0
\end{aligned} \tag{22}$$

where

$$T_{11}^{(i)}(\alpha r) = (n^2 + n - \frac{1}{2}\beta^2 r^2)C_n^{(i)}(\alpha r) - \alpha r C_{n-1}^{(i)}(\alpha r)$$

$$T_{12}^{(i)}(\beta r) = \mp[-(n^2+n)C_n^{(i)}(\beta r) + n\beta r C_{n-1}^{(i)}(\beta r)]$$

$$T_{21}^{(i)}(\alpha r) = -(n^2+n-\frac{1}{2}\beta^2 r^2 - \alpha^2 r^2)C_n^{(i)}(\alpha r) + \alpha r C_{n-1}^{(i)}(\alpha r)$$

$$T_{22}^{(i)}(\beta r) = \mp[(n^2+n)C_n^{(i)}(\beta r) - n\beta r C_{n-1}^{(i)}(\beta r)] \tag{23}$$

$$T_{41}^{(i)}(\alpha r) = \pm[-(n^2+n)C_n^{(i)}(\alpha r) + n\alpha r C_{n-1}^{(i)}(\alpha r)]$$

$$T_{42}^{(i)}(\beta r) = -(n^2+n-\frac{1}{2}\beta^2 r^2)C_n^{(i)}(\beta r) + \beta r C_{n-1}^{(i)}(\beta r)$$

$$T_{51}^{(i)}(\alpha r) = (\alpha^2 r^2 - \frac{1}{2}\beta^2 r^2)C_n^{(i)}(\alpha r).$$

1.2 Frequency Equations

Boundary conditions for free vibrations are:

$$\sigma_r = \sigma_{r\theta} = 0 \quad \text{on} \quad \mathrm{r = a}. \tag{24}$$

We have

$$\sigma_r = \frac{2\mu}{a^2}\sum_{n=0}^{N}\sum_{k=1}^{L}[A_{nk}T_{11}^{(3)}(\alpha_{nk}a) + iB_{nk}T_{12}^{(3)}(\beta_{nk}a)]e^{in\theta} \tag{25a}$$

$$\sigma_{r\theta} = \frac{2\mu}{a^2}\sum_{n=0}^{N}\sum_{k=1}^{L}[iA_{nk}T_{41}^{(3)}(\alpha_{nk}a) + B_{nk}T_{42}^{(3)}(\beta_{nk}a)]e^{in\theta}. \tag{25b}$$

Setting σ_r and $\sigma_{r\theta}$ equal to zero at r = a, we obtain a set of two homogeneous equations. The determinant of this set must vanish for the existence of a non-trivial solution for A_{nk} and B_{nk}. Thus we have the frequency equation:

$$\Delta = \begin{vmatrix} T_{11}^{(3)}(\alpha_{nk}a) & iT_{12}^{(3)}(\beta_{nk}a) \\ iT_{41}^{(3)}(\alpha_{nk}a) & T_{42}^{(3)}(\beta_{nk}a) \end{vmatrix} = 0. \tag{26}$$

$$T_{11}^{(3)}(\alpha_{nk}a)T_{42}^{(3)}(\beta_{nk}a) + T_{41}^{(3)}(\alpha_{nk}a)T_{12}^{(3)}(\beta_{nk}a) = 0 \tag{27}$$

where

$$T_{11}^{(3)}(\alpha a) = (n^2+n-\frac{1}{2}\beta^2 a^2)H_n^{(1)}(\alpha a) - \alpha a H_{n-1}^{(1)}(\alpha a)$$

$$T_{12}^{(3)}(\beta a) = -(n^2+n)H_n^{(1)}(\beta a) + n\beta a H_{n-1}^{(1)}(\beta a) \tag{28}$$

$$T_{41}^{(3)}(\alpha a) = -(n^2+n)H_n^{(1)}(\alpha a) + n\alpha a H_{n-1}^{(1)}(\alpha a)$$

$$T_{42}^{(3)}(\beta a) = -(n^2 + n - \frac{1}{2}\beta^2 a^2) H_n^{(1)}(\beta a) + \beta a H_{n-1}^{(1)}(\beta a).$$

1.2.1. Axially Symmetric Case: n = 0

We have:

$$\begin{aligned} T_{11}^{(3)}(\alpha a) &= -\frac{1}{2}\beta^2 a^2 H_0^{(1)}(\alpha a) + \alpha a H_1^{(1)}(\alpha a) \\ T_{12}^{(3)}(\beta a) &= 0 \\ T_{41}^{(3)}(\alpha a) &= 0 \\ T_{42}^{(3)}(\beta a) &= \frac{1}{2}\beta^2 a^2 H_0^{(1)}(\beta a) - \beta a H_1^{(1)}(\beta a). \end{aligned} \tag{29}$$

Then the frequency equation becomes

$$T_{11}^{(3)}(\alpha_{nk} a) T_{42}^{(3)}(\beta_{nk} a) = 0. \tag{30}$$

For the given Poisson's ratio, ν, frequency ratio k is fixed, i.e.:

$$k = \frac{\alpha^2}{\beta^2} = \frac{\mu}{\lambda + 2\mu} = \frac{1 - 2\nu}{2(1-\nu)} \tag{31}$$

and so,

$$\alpha = \sqrt{\frac{1-2\nu}{2(1-\nu)}}\beta. \tag{32}$$

If $\nu = \frac{1}{4}$, then $k = \frac{1}{3}$.

i.e. $\beta = \sqrt{3}\alpha$

Then the frequency equation becomes:

$$[H_0^{(1)}(\alpha_{0k} a) - \frac{2}{3\alpha_{0k} a} H_1^{(1)}(\alpha_{0k} a)][H_0^{(1)}(\sqrt{3}\alpha_{0k} a) - \frac{2}{\sqrt{3}\alpha_{0k} a} H_1^{(1)}(\sqrt{3}\alpha_{0k} a)] = 0. \tag{33}$$

Eqn. (33) can be solved for the roots α_{0k}, which, in turn, yield ω_{0k}, if a, ν and c_1 are known.

1.2.2. Nonaxially Symmetric Case: $n \neq 0$

The frequency equation becomes, if $\nu = \frac{1}{4}$,

$$T_{11}^{(3)}(\alpha_{nk}a)T_{42}^{(3)}(\sqrt{3}\alpha_{nk}a) + T_{41}^{(3)}(\alpha_{nk}a)T_{12}^{(3)}(\sqrt{3}\alpha_{nk}a) = 0 \tag{34}$$

where, n = 1,2,...,N and k = 0,1,2,...,K.

We have K number of frequency, α_{nk}, for each n.

1.3 Least Square Method

Unknown constants A_{nk} and B_{nk} can be determined from the initial conditions at t = 0. For the constant velocity initial conditions,

$$\begin{aligned} \dot{u}_r(a,0,0) &= \overline{V}_r = V_0 \\ \dot{u}_\theta(a,0,0) &= \overline{V}_\theta = 0. \end{aligned} \tag{35}$$

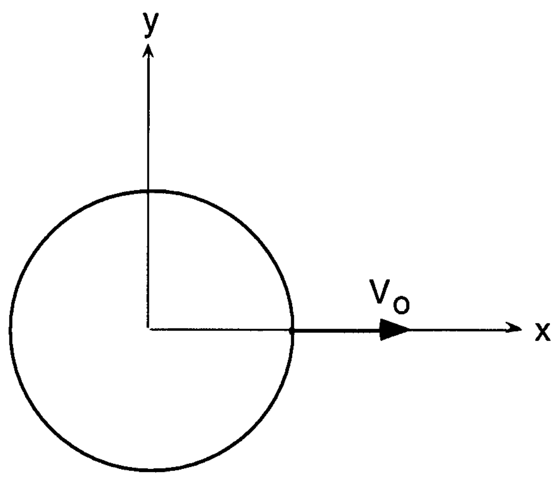

Figure 2. Velocity initial condition.

From Eqns. (18), radial and circumferential velocities are the real part of:

$$\frac{\partial}{\partial t}u_r = \frac{1}{r}\sum_{n=0}^{N}\sum_{k=1}^{L}[-iA_{nk}\omega_{nk}U_1^{(3)}(\alpha_{nk}r) - B_{nk}\omega_{nk}U_2^{(3)}(\beta_{nk}r)]e^{in\theta}e^{-i\omega t} \tag{36a}$$

$$\frac{\partial}{\partial t}u_\theta = \frac{1}{r}\sum_{n=0}^{N}\sum_{k=1}^{L}[A_{nk}\omega_{nk}V_1^{(3)}(\alpha_{nk}r) - iB_{nk}\omega_{nk}V_2^{(3)}(\beta_{nk}r)]e^{in\theta}e^{-i\omega t} \tag{36b}$$

Then

$$\mathrm{Re}\,V_r = \frac{1}{r}\sum_{n=0}^{N}\sum_{k=1}^{L}[A_{nk}\omega_{nk}U_1^{(3)}(\alpha_{nk}r)\sin n\theta - B_{nk}\omega_{nk}U_2^{(3)}(\beta_{nk}r)\cos n\theta] \tag{37a}$$

$$\mathrm{Re}\,V_\theta = \frac{1}{r}\sum_{n=0}^{N}\sum_{k=1}^{L}[A_{nk}\omega_{nk}V_1^{(3)}(\alpha_{nk}r)\cos n\theta + B_{nk}\omega_{nk}V_2^{(3)}(\beta_{nk}r)\sin n\theta]. \tag{37b}$$

The problem is to minimize the functional:

$$F(A_{nk},B_{nk}) = \sum_{j=1}^{m}[\mathrm{Re}\,V_r(r_j,\theta_j A_{nk},B_{nk}) - \overline{V}_r]^2 + \sum_{j=1}^{m}[\mathrm{Re}\,V_\theta(r_j,\theta_j A_{nk},B_{nk}) - \overline{V}_\theta]^2. \tag{38}$$

The first terms in each bracket are the actual velocities from the elastic solutions and the second terms are the specified velocities from initial conditions.

We have:

$$\begin{aligned}
&\frac{\partial F}{\partial A_{i1}} = 0, \frac{\partial F}{\partial A_{i2}} = 0, \ldots, \frac{\partial F}{\partial A_{iL}} = 0 \\
&\frac{\partial F}{\partial B_{i1}} = 0, \frac{\partial F}{\partial B_{i2}} = 0, \ldots, \frac{\partial F}{\partial B_{iL}} = 0
\end{aligned} \tag{39}$$

where $i = 0,1,2,\ldots,N$.

The final matrix form of Eqn. (38) is obtained as follows by Least-Squares procedures. Matrix formulations are shown in Appendix C:

$$\begin{bmatrix} \mathbf{P}^T\mathbf{P}+\mathbf{Q}^T\mathbf{Q} & \mathbf{P}^T\mathbf{C}+\mathbf{Q}^T\mathbf{D} \\ \mathbf{C}^T\mathbf{P}+\mathbf{D}^T\mathbf{Q} & \mathbf{C}^T\mathbf{C}+\mathbf{D}^T\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^T & \mathbf{Q}^T \\ \mathbf{C}^T & \mathbf{D}^T \end{bmatrix} \begin{bmatrix} \mathbf{S} \\ \mathbf{T} \end{bmatrix} \tag{40}$$

where

$$\begin{bmatrix} A_{01} \\ A_{02} \\ \\ A_{0l} \\ \vdots \\ A_{n1} \\ A_{n2} \\ \\ A_{nl} \end{bmatrix} = \mathbf{A} \quad \text{and} \quad \begin{bmatrix} B_{01} \\ B_{02} \\ \\ B_{0l} \\ \vdots \\ B_{n1} \\ B_{n2} \\ \\ B_{nl} \end{bmatrix} = \mathbf{B} \tag{41}$$

$$\begin{bmatrix} \overline{V}_{r_1} \\ \overline{V}_{r_2} \\ \overline{V}_{r_3} \\ \vdots \\ \overline{V}_{r_m} \end{bmatrix} = \mathbf{S} \text{ and } \begin{bmatrix} \overline{V}_{\theta_1} \\ \overline{V}_{\theta_2} \\ \overline{V}_{\theta_3} \\ \vdots \\ \overline{V}_{\theta_m} \end{bmatrix} = \mathbf{T} \tag{42}$$

$$\mathbf{P} = \begin{bmatrix} p_{1,01} & \cdots & p_{1,0l} & p_{1,11} & \cdots & p_{1,1l} & \cdots & p_{1,n1} & \cdots & p_{1,nl} \\ p_{2,01} & \cdots & p_{2,0l} & p_{2,11} & \cdots & p_{2,1l} & \cdots & p_{2,n1} & \cdots & p_{2,nl} \\ \\ \\ p_{m,01} & \cdots & p_{m,0l} & p_{m,11} & \cdots & p_{m,1l} & \cdots & p_{m,n1} & \cdots & p_{m,nl} \end{bmatrix} \tag{43a}$$

$$\mathbf{Q} = \begin{bmatrix} q_{1,01} & \cdots & q_{1,0l} & q_{1,11} & \cdots & q_{1,1l} & \cdots & q_{1,n1} & \cdots & q_{1,nl} \\ q_{2,01} & \cdots & q_{2,0l} & q_{2,11} & \cdots & q_{2,1l} & \cdots & q_{2,n1} & \cdots & q_{2,nl} \\ \\ \\ q_{m,01} & \cdots & q_{m,0l} & q_{m,11} & \cdots & q_{m,1l} & \cdots & q_{m,n1} & \cdots & q_{m,nl} \end{bmatrix}. \tag{43b}$$

Matrix $\mathbf{C}$ and $\mathbf{D}$ have the same pattern as $\mathbf{P}$ and $\mathbf{Q}$.

The components of the matrix $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{C}$ and $\mathbf{D}$ are:

$$p_{j,ik} = \frac{1}{r_j}\omega_{ik} U_1^{(3)}(\alpha_{ik} r_j) \sin i\theta_j \tag{44a}$$

$$q_{j,ik} = \frac{1}{r_j}\omega_{ik} V_1^{(3)}(\alpha_{ik} r_j) \cos i\theta_j \tag{44b}$$

$$c_{j,ik} = -\frac{1}{r_j}\omega_{ik} U_2^{(3)}(\beta_{ik} r_j) \cos i\theta_j \tag{44c}$$

$$d_{j,ik} = \frac{1}{r_j}\omega_{ik} V_2^{(3)}(\beta_{ik} r_j) \sin i\theta_j. \tag{44d}$$

1.4 Fourier Transform Method

Another available method to solve the problems other than using the Least square procedure is the Fourier transform technique. Since Hankel functions are complex valued there is no point in insisting upon the real form of Fourier series. Thus we write for the Fourier transform of the displacement and stress fields:

$$\bar{u}_r = \frac{1}{r}\sum_{n=-\infty}^{\infty}[A_n(\omega)U_1^{(3)}(\alpha r) + B_n(\omega)U_2^{(3)}(\beta r)]e^{in\theta}$$
$$\bar{u}_\theta = \frac{1}{r}\sum_{n=-\infty}^{\infty}[iA_n(\omega)V_1^{(3)}(\alpha r) - iB_n(\omega)V_2^{(3)}(\beta r)]e^{in\theta} \tag{45}$$

and

$$\bar{\sigma}_r = \frac{2\mu}{r^2}\sum_{n=-\infty}^{\infty}[A_n(\omega)T_{11}^{(3)}(\alpha r) + B_n(\omega)T_{12}^{(3)}(\beta r)]e^{in\theta}$$
$$\bar{\sigma}_\theta = \frac{2\mu}{r^2}\sum_{n=-\infty}^{\infty}[A_n(\omega)T_{21}^{(3)}(\alpha r) + B_n(\omega)T_{22}^{(3)}(\beta r)]e^{in\theta}$$
$$\bar{\sigma}_{r\theta} = \frac{2\mu}{r^2}\sum_{n=-\infty}^{\infty}[iA_n(\omega)T_{41}^{(3)}(\alpha r) - iB_n(\omega)T_{42}^{(3)}(\beta r)]e^{in\theta} \tag{46}$$

where $U_1^{(3)}(\alpha r)$, ... , $T_{42}^{(3)}(\beta r)$ are the same functions defined by Eqns. (19) and (23) with a positive sign only, in which $C_n^{(3)} = H_n^{(1)}$ is used for $C_n^{(i)}$. The conditions $A_n(\omega)$ and $B_n(\omega)$ are to be determined from the boundary conditions.

1.4.1. Boundary Conditions

Surface velocities are assumed to be given on the surface of the cavity r = a; i.e.,

$$\dot{u}_r(a,\theta,t) = V_r(\theta,t) \tag{47a}$$

$$\dot{u}_\theta(a,\theta,t) = V_\theta(\theta,t). \tag{47b}$$

Expanding the Fourier transforms of these into Fourier series we have:

$$\bar{V}_r(\theta,\omega) = \sum_{n=-\infty}^{\infty} T_n(\omega)e^{in\theta} \tag{48a}$$

$$\bar{V}_\theta(\theta,\omega) = \sum_{n=-\infty}^{\infty} S_n(\omega)e^{in\theta} \tag{48b}$$

where $T_n(\omega)$ and $S_n(\omega)$ are determined through:

$$\begin{Bmatrix} T_n(\omega) \\ S_n(\omega) \end{Bmatrix} = \frac{1}{2\pi}\int_0^{2\pi}\begin{Bmatrix} \bar{V}_r(\theta,\omega) \\ \bar{V}_\theta(\theta,\omega) \end{Bmatrix} e^{-in\theta}d\theta \tag{49}$$

where $n = 0, \pm 1, \pm 2, \ldots$

Since velocity becomes:

$$\overline{V}_r(\theta,\omega)=\frac{1}{a}\sum_{n=-\infty}^{\infty}[-i\omega A_n(\omega)U_1^{(3)}(\alpha a)-i\omega B_n(\omega)U_2^{(3)}(\beta a)]e^{in\theta}$$

$$\overline{V}_\theta(\theta,\omega)=\frac{1}{a}\sum_{n=-\infty}^{\infty}[\omega A_n(\omega)V_1^{(3)}(\alpha a)-\omega B_n(\omega)V_2^{(3)}(\beta a)]e^{in\theta}, \tag{50}$$

the boundary conditions from Equation (47) now give:

$$\begin{bmatrix} U_1^{(3)}(\alpha a) & U_2^{(3)}(\beta a) \\ V_1^{(3)}(\alpha a) & V_2^{(3)}(\beta a) \end{bmatrix}\begin{Bmatrix} A_n(\omega) \\ B_n(\omega) \end{Bmatrix}=(\frac{a}{\omega})\begin{bmatrix} iT_n(\omega) \\ S_n(\omega) \end{bmatrix}. \tag{51}$$

The solution of this set is

$$A_n(\omega)=(\frac{a}{\omega})[-iT_n(\omega)V_2^{(3)}(\beta a)-S_n(\omega)U_2^{(3)}(\beta a)]/\det \tag{52a}$$

$$B_n(\omega)=(\frac{a}{\omega})[S_n(\omega)U_1^{(3)}(\alpha a)-iT_n(\omega)V_1^{(3)}(\alpha a)]/\det \tag{52b}$$

where

$$\det=-U_1^{(3)}(\alpha a)V_2^{(3)}(\beta a)-V_1^{(3)}(\alpha a)U_2^{(3)}(\beta a). \tag{53}$$

1.4.2. Displacements and Stresses

Substituting the above $A_n(\omega)$ and $B_n(\omega)$ into Eqns. (45) and (46) we obtain the solutions as follows. Displacement fields are:

$$\overline{u}_r(r,\theta,\omega)=\frac{1}{r}\sum_{n=-\infty}^{\infty}[iT_n(\omega)u_n^{(1)}(\omega r)+S_n(\omega)u_n^{(2)}(\omega r)]e^{in\theta} \tag{54a}$$

$$\overline{u}_\theta(r,\theta,\omega)=\frac{1}{r}\sum_{n=-\infty}^{\infty}[T_n(\omega)\upsilon_n^{(1)}(\omega r)+iS_n(\omega)\upsilon_n^{(2)}(\omega r)]e^{in\theta} \tag{54b}$$

where

$$u_n^{(1)}(\omega r)=(\frac{a}{\omega})[-V_2^{(3)}(\beta a)U_1^{(3)}(\alpha r)-V_1^{(3)}(\alpha a)U_2^{(3)}(\beta r)]\Big/\det$$

$$u_n^{(2)}(\omega r)=(\frac{a}{\omega})[-U_2^{(3)}(\beta a)U_1^{(3)}(\alpha r)+U_1^{(3)}(\alpha a)U_2^{(3)}(\beta r)]\Big/\det$$

$$\upsilon_n^{(1)}(\omega r)=(\frac{a}{\omega})[V_2^{(3)}(\beta a)V_1^{(3)}(\alpha r)-V_1^{(3)}(\alpha a)V_2^{(3)}(\beta r)]\Big/\det \tag{55}$$

$$\upsilon_n^{(2)}(\omega r)=(\frac{a}{\omega})[-U_2^{(3)}(\beta a)V_1^{(3)}(\alpha r)-U_1^{(3)}(\alpha a)V_2^{(3)}(\beta r)]\Big/\det.$$

Stress fields are:

$$\overline{\sigma}_r(r,\theta,\omega) = \frac{2\mu}{r^2}\sum_{n=-\infty}^{\infty}[iT_n(\omega)t_n^{(1)}(\omega r) + S_n(\omega)t_n^{(2)}(\omega r)]e^{in\theta} \tag{56a}$$

$$\overline{\sigma}_\theta(r,\theta,\omega) = \frac{2\mu}{r^2}\sum_{n=-\infty}^{\infty}[iT_n(\omega)s_n^{(1)}(\omega r) + S_n(\omega)s_n^{(2)}(\omega r)]e^{in\theta} \tag{56b}$$

$$\overline{\sigma}_{r\theta}(r,\theta,\omega) = \frac{2\mu}{r^2}\sum_{n=-\infty}^{\infty}[T_n(\omega)\tau_n^{(1)}(\omega r) + iS_n(\omega)\tau_n^{(2)}(\omega r)]e^{in\theta} \tag{56c}$$

where

$$\begin{aligned}
t_n^{(1)}(\omega r) &= (\frac{a}{\omega})[-V_2^{(3)}(\beta a)T_{11}^{(3)}(\alpha r) - V_1^{(3)}(\alpha a)T_{12}^{(3)}(\beta r)]\Big/\text{det} \\
t_n^{(2)}(\omega r) &= (\frac{a}{\omega})[-U_2^{(3)}(\beta a)T_{11}^{(3)}(\alpha r) + U_1^{(3)}(\alpha a)T_{12}^{(3)}(\beta r)]\Big/\text{det} \\
s_n^{(1)}(\omega r) &= (\frac{a}{\omega})[-V_2^{(3)}(\beta a)T_{21}^{(3)}(\alpha r) - V_1^{(3)}(\alpha a)T_{22}^{(3)}(\beta r)]\Big/\text{det} \\
s_n^{(2)}(\omega r) &= (\frac{a}{\omega})[-U_2^{(3)}(\beta a)T_{21}^{(3)}(\alpha r) + U_1^{(3)}(\alpha a)T_{22}^{(3)}(\beta r)]\Big/\text{det} \\
\tau_n^{(1)}(\omega r) &= (\frac{a}{\omega})[V_2^{(3)}(\beta a)T_{41}^{(3)}(\alpha r) - V_1^{(3)}(\alpha a)T_{42}^{(3)}(\beta r)]\Big/\text{det} \\
\tau_n^{(2)}(\omega r) &= (\frac{a}{\omega})[-U_2^{(3)}(\beta a)T_{41}^{(3)}(\alpha r) - U_1^{(3)}(\alpha a)T_{42}^{(3)}(\beta r)]\Big/\text{det}.
\end{aligned} \tag{57}$$

Velocity Boundary Conditions

When we have the boundary conditions as

$$\text{(i)} \quad V_r(a,\theta,t) = V_r(\theta,t) = V_0 \quad \text{for} \quad -\theta_0 \le \theta \le \theta_0 \\ = 0 \quad \text{for elsewhere} \tag{58a}$$

and

$$\text{(ii)} \quad V_\theta(a,\theta,t) = 0, \tag{58b}$$

then

$$T_n(\omega) = \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} V_0 e^{-in\theta}d\theta$$

$$= \frac{V_0}{n\pi}\sin(n\theta_0) \tag{59a}$$

$$S_n(\omega) = 0. \tag{59b}$$

We obtain the displacement and stress fields by substituting $T_n(\omega)$ and $S_n(\omega)$ into Eqns. (54) and (56) as follows:

$$\begin{aligned}
\bar{u}_r(r,\theta,\omega) &= \frac{1}{r}\sum_{n=-\infty}^{\infty} i\frac{V_0 \sin n\theta_0}{n\pi}\left(\frac{a}{\omega}\right)\frac{V_2^{(3)}(\beta a)U_1^{(3)}(\alpha r) + V_1^{(3)}(\alpha a)U_2^{(3)}(\beta r)}{U_1^{(3)}(\alpha a)V_2^{(3)}(\beta a) + V_1^{(3)}(\alpha a)U_2^{(3)}(\beta a)} e^{in\theta} \\
\bar{u}_\theta(r,\theta,\omega) &= \frac{1}{r}\sum_{n=-\infty}^{\infty} \frac{V_0 \sin n\theta_0}{n\pi}\left(\frac{a}{\omega}\right)\frac{-V_2^{(3)}(\beta a)V_1^{(3)}(\alpha r) + V_1^{(3)}(\alpha a)V_2^{(3)}(\beta r)}{U_1^{(3)}(\alpha a)V_2^{(3)}(\beta a) + V_1^{(3)}(\alpha a)U_2^{(3)}(\beta a)} e^{in\theta}
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
\bar{\sigma}_r(r,\theta,\omega) &= \frac{2\mu}{r^2}\sum_{n=-\infty}^{\infty} i\frac{V_0 \sin n\theta_0}{n\pi}\left(\frac{a}{\omega}\right)\frac{V_2^{(3)}(\beta a)T_{11}^{(3)}(\alpha r) + V_1^{(3)}(\alpha a)T_{12}^{(3)}(\beta r)}{U_1^{(3)}(\alpha a)V_2^{(3)}(\beta a) + V_1^{(3)}(\alpha a)U_2^{(3)}(\beta a)} e^{in\theta} \\
\bar{\sigma}_\theta(r,\theta,\omega) &= \frac{2\mu}{r^2}\sum_{n=-\infty}^{\infty} i\frac{V_0 \sin n\theta_0}{n\pi}\left(\frac{a}{\omega}\right)\frac{V_2^{(3)}(\beta a)T_{21}^{(3)}(\alpha r) + V_1^{(3)}(\alpha a)T_{22}^{(3)}(\beta r)}{U_1^{(3)}(\alpha a)V_2^{(3)}(\beta a) + V_1^{(3)}(\alpha a)U_2^{(3)}(\beta a)} e^{in\theta} \\
\bar{\sigma}_{r\theta}(r,\theta,\omega) &= \frac{2\mu}{r^2}\sum_{n=-\infty}^{\infty} \frac{V_0 \sin n\theta_0}{n\pi}\left(\frac{a}{\omega}\right)\frac{-V_2^{(3)}(\beta a)T_{41}^{(3)}(\alpha r) + V_1^{(3)}(\alpha a)T_{42}^{(3)}(\beta r)}{U_1^{(3)}(\alpha a)V_2^{(3)}(\beta a) + V_1^{(3)}(\alpha a)U_2^{(3)}(\beta a)} e^{in\theta}
\end{aligned} \tag{61}$$

where $U_1^{(3)}(\alpha r),\ldots.,T_{42}^{(3)}(\beta r)$ are the same functions defined by Eqns. (19) and (23) with positive signs only, in which $C_n^{(3)} = H_n^{(1)}$ is used for $C_n^{(i)}$. Now the inverse of Fourier transforms of these equations gives the complete solutions.

2.0 Elastic-Plastic Boundary

The main problem is to determine the contour C that separates the plastic from the elastic region such that the displacements, as well as stresses, are continuous throughout the exterior of the cavity surface. If this elastic-plastic boundary C has been found, the problem divides into a pure plastic problem and a pure elastic one so that the elastic and plastic solutions can be obtained for each time step.

We have three cases as we evaluate the Von-Mises yield condition along the each sector:

(Case 1) $\frac{3}{2}(\sigma_r - \sigma_\theta)^2 + 6\sigma_{r\theta}{}^2 < 2Y^2$: elastic

(Case 2) $\frac{3}{2}(\sigma_r - \sigma_\theta)^2 + 6\sigma_{r\theta}{}^2 = 2Y^2$: elastic-plastic boundary

(Case 3) $\frac{3}{2}(\sigma_r - \sigma_\theta)^2 + 6\sigma_{r\theta}{}^2 > 2Y^2$: plastic.

As we keep checking the yield condition exterior of the cavity surface, the elastic-plastic boundary has been found if the solutions at the node point become elastic. Then the stresses inside the boundary C are plastic and can be obtained by using the equilibrium equations and yield condition.

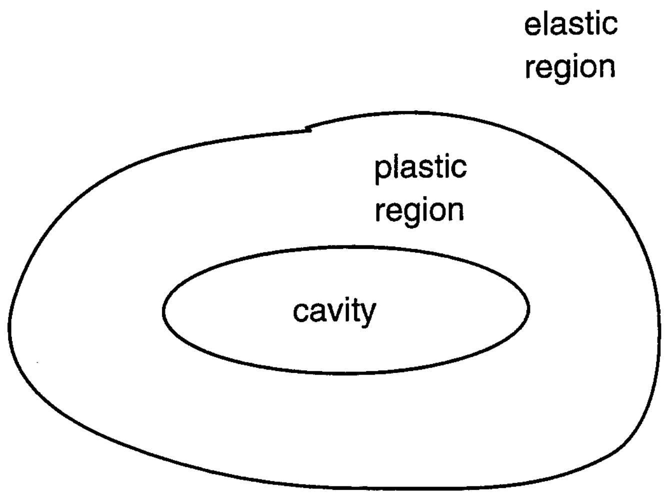

Figure 3. Elastic and plastic regions.

3.0 Plastic Region

3.1 Stresses

We have the equilibrium equations, in which the body forces are neglected, for the plane strain case in the cylindrical coordinates as follows:

$$\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\sigma_r-\sigma_\theta}{r}=0 \tag{62a}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}+\frac{2}{r}\sigma_{r\theta}=0. \tag{62b}$$

At the elastic-plastic boundary, plastic solutions are the same as elastic solutions because of the continuity condition. Plastic solutions at the j = s node on contour Γ_s now can be found with the stresses at the previous contour Γ_{s+1} from the elastic-plastic boundary up to the cavity wall.

$$\frac{(\sigma_r)_{j^{s+1}}-(\sigma_r)_{j^s}}{\Delta r}+\frac{1}{r_{j^{s+1}}}\frac{(\sigma_{r\theta})_{j^{s+1}+1}-(\sigma_{r\theta})_{j^{s+1}}}{\Delta\theta}+\frac{(\sigma_r)_{j^{s+1}}-(\sigma_\theta)_{j^{s+1}}}{r_{j^{s+1}}}=0 \tag{63a}$$

$$\frac{(\sigma_{r\theta})_{j^{s+1}}-(\sigma_{r\theta})_{j^s}}{\Delta r}+\frac{1}{r_{j^{s+1}}}\frac{(\sigma_\theta)_{j^{s+1}+1}-(\sigma_\theta)_{j^{s+1}}}{\Delta\theta}+\frac{2}{r_{j^{s+1}}}(\sigma_{r\theta})_{j^{s+1}}=0 \tag{63b}$$

Then we have three components of stresses as follows:

$$(\sigma_r)_{j^s} = (\sigma_r)_{j^{s+1}} + \frac{\Delta r}{r_{j^{s+1}}\Delta\theta}[(\sigma_{r\theta})_{j^{s+1}+1} - (\sigma_{r\theta})_{j^{s+1}}] + \frac{\Delta r}{r_{j^{s+1}}}[(\sigma_r)_{j^{s+1}} - (\sigma_\theta)_{j^{s+1}}] \tag{64a}$$

$$(\sigma_{r\theta})_{j^s} = (\sigma_{r\theta})_{j^{s+1}} + \frac{\Delta r}{r_{j^{s+1}}\Delta\theta}[(\sigma_\theta)_{j^{s+1}+1} - (\sigma_\theta)_{j^{s+1}}] + \frac{2\Delta r}{r_{j^{s+1}}}(\sigma_{r\theta})_{j^{s+1}} \tag{64b}$$

$$(\sigma_\theta)_{j^s} = (\sigma_r)_{j^s} + \sqrt{\frac{4}{3}Y^2 - 4(\sigma_{r\theta})^2_{j^s}}\,. \tag{64c}$$

3.2 Displacements

In the plastic region, the displacements can be determined from the fact that $\varepsilon_r + \varepsilon_\theta$ is still given by Hooke's law, since the sum of the plastic part of ε_r and ε_θ vanishes. Assuming small strains, the displacements equations become:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial \upsilon}{\partial \theta} = \frac{1-2\nu}{2G}(\sigma_r + \sigma_\theta) \tag{65a}$$

and

$$\frac{\partial \upsilon}{\partial r} - \frac{\upsilon}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\sigma_{r\theta}}{G}. \tag{65b}$$

Then we have:

$$\frac{u_{j^{s+1}} - u_{j^s}}{\Delta r} + \frac{\upsilon_{j^{s+1}+1} - \upsilon_{j^{s+1}}}{r_{j^{s+1}}\Delta\theta} + \frac{u_{j^{s+1}}}{r_{j^{s+1}}} = \frac{1-2\nu}{2G}\{(\sigma_r)_{j^{s+1}} - (\sigma_\theta)_{j^{s+1}}\} \tag{66a}$$

$$\frac{\upsilon_{j^{s+1}} - \upsilon_{j^s}}{\Delta r} + \frac{u_{j^{s+1}+1} - u_{j^{s+1}}}{r_{j^{s+1}}\Delta\theta} - \frac{\upsilon_{j^{s+1}}}{r_{j^{s+1}}} = \frac{(\sigma_{r\theta})_{j^{s+1}}}{\mu}. \tag{66b}$$

Now we have two components of displacements as follows:

$$\frac{1}{\Delta r}u_{j^s} = \frac{1}{\Delta r}u_{j^{s+1}} + \frac{\upsilon_{j^{s+1}+1} - \upsilon_{j^{s+1}}}{r_{j^{s+1}}\Delta\theta} + \frac{u_{j^{s+1}}}{r_{j^{s+1}}} - \frac{1-2\nu}{2G}\{(\sigma_r)_{j^{s+1}} - (\sigma_\theta)_{j^{s+1}}\} \tag{67a}$$

$$\frac{1}{\Delta r}\upsilon_{j^s} = \frac{1}{\Delta r}\upsilon_{j^{s+1}} + \frac{u_{j^{s+1}+1} - u_{j^{s+1}}}{r_{j^{s+1}}\Delta\theta} - \frac{\upsilon_{j^{s+1}}}{r_{j^{s+1}}} - \frac{(\sigma_{r\theta})_{j^{s+1}}}{\mu}. \tag{67b}$$

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References

[1] Woo, H. J. and Partom, Y., "Cavity Expansion Resistance Model for Arbitrary Cross Sections: Part I," Institute for Advanced Technology Report, IAT.R 0093, October 1995.

[2] Woo, H. J., "Cavity Expansion Model for Arbitrary Cross Sections: Part II," Institute for Advanced Technology Report, September 1996.

[3] Bishop, R. F., Hill, R., and Mott, N. F., "The Theory of Indentation and Hardness Tests," *The Proceedings of the Physical Society*, Vol. 57, Part 3, pp. 147-159, 1945.

[4] Chakrabarty, J., *Theory of Plasticity,* McGraw-Hill Book Company, 1987.

[5] Hill, R., *The Mathematical Theory of Plasticity*, Oxford University Press, London, 1950.

[6] Lubliner, J., *Plasticity Theory*, Macmillan Publishing Company, 1990.

[7] Partom, Y., "Static Cavity Expansion Model for Partially Confined Targets," Institute for Advanced Technology Report, IAT.R 0092, January 1996.

[8] Partom, Y., "Efficiency of Lateral Self Confinement in Metal and Ceramic Targets," Institute for Advanced Technology Report, IAT.R 0019, April 1993.

[9] Gazis, D. C., "Exact Analysis of the Plane-Strain Vibrations of Thick-Walled Hollow Cylinders," *The Journal of the Acoustical Society of America*, Vol. 30., No. 8., pp. 786-794, Aug, 1958.

[10] Gazis, D. C., "Three-Dimensional Investigation of the Propagation of Waves in Hollow Circular Cylinders. I. Analytical Foundation, II. Numerical Results," *The Journal of the Acoustical Society of America*, Vol. 31., No. 5., pp. 568-578, May, 1959.

[11] Armenakas, A. E., Gazis, D. C. and Herrmann, G. H., *Free Vibrations of Circular Cylindrical Shells*, Pergamon Press Ltd., Headington Hill Hall, Oxford, 1969.

[12] Proakis, J. G. and Manolakis, D. G., *Digital Signal Processing: Principles, Algorithms, and Applications*, 2nd Ed., Macmillan Publishing Company, 1992.

[13] Eringen, A. C., "Response of an Elastic Disk to Impact and Moving Loads," *Quart. J. Appl. Mech.*, Vol. 8, pp. 385-393, 1955.

[14] Eringen, A. C., "Elasto-Dynamic Problem Concerning the Spherical Cavity," *Quart. J. Mech. Appl. Math.*, Vol. 10, pp. 257-270, 1957.

[15] Liu, C. K. and Lee, T. N., "Dynamic Response of an Infinite Cylinder to Asymmetric Pressure on Its Lateral Surface," *Development in Theoretical and Applied Mechanics*, Vol. 3, pp. 447-464, Pergamon Press, 1967

[16] Forrestal, M. J. and Luk, V. K., "Dynamic Spherical Cavity-Expansion in a Compressible Elastic-Plastic Solid," *J. Appl. Mech.*, Vol. 55, pp. 275-279, June, 1988.

[17] Forrestal, M. J., Tzou, D. Y., Askari, E. and Longcope, D. B., "Penetration into Ductile Metal Targets with Rigid Spherical Nose Rods," to appear *Int. J. Impact Eng.*

[18] Nottrot, R. and Timman, R., "A General Method of Solving the Plane Elasto-Plastic Problems," *Journal of Engineering Mathematics*, Vol. 1, pp. 19-36, 1967.

[19] Banichuk, N. V., "Calculating the Loading on an Elastoplastic Body," *Mechanics of Solids*, Vol. 9, pp. 117-124, 1969.

[20] Annin, B. D. and Cherepanov, G. P., *Elastic-Plastic Problems*, ASME Press, New York, 1988.

[21] Vodicka, V., "Elastic Waves in an Infinite Medium in the Case of Plane Strain," *Appl. Sci. Res.*, Section A, Vol. 10, pp. 235-240, 1961.

[22] Gazis, D. C. and Mindlin, R. D., "Extensional Vibrations and Waves in a Circular Disk and a Semi-Infinite Plate," *J. Appl. Mech.*, Vol. 27, pp. 541-547, September, 1960.

[23] Miklowitz, J., "Plane-Stress Unloading Waves Emanating from a Suddenly Punched Hole in a Stretched Elastic Plate," *J. Appl. Mech.*, pp. 165-171, March, 1960.

[24] Kinsler, L. E., Frey, A. R., Coppens, A. B. and Sanders, J. V., *Fundamentals of Acoustics*, John Wiley & Sons, 3rd Ed., 1982.

[25] James, M. L., Smith, G. M. and Wolford, J. C., *Applied Numerical Methods for Digital Computation With FORTRAN and CSMP*, Harper & Row Publishers, Inc., 2nd Ed., 1977.

[26] Press, W. H., Flannery, B. P., Teukolsky, S. A. and Vetterling, W. T., *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, 1986.

[27] Shaffer, B. W. and House, R. N., "Displacements in a Wide Curved Bar Subjected to Pure Elastic-Plastic Bending," *J. Appl. Mech.*, Vol. 26, p. 447, 1957.

[28] Shaffer, B. W. and House, R. N., "The Elastic-Plastic Stress Distribution Within a Wide Curved Bar Subjected to Pure Bending," *J. Appl. Mech.*, Vol. 24, p. 305, 1955.

[29] Eason, G., "The Elastic-Plastic Bending of A Compressible Curved Bar," *Appl. Sci. Res.*, A, Vol. 9, p. 53, 1960.

[30] Forsythe, G. E. and Wasow, W. R., *Finite Difference Methods for Partial Differential Equations*, John Wiley & Sons, Inc., 1960.

[31] Alder, B., Fernbach, S. and Rotenberg, M., *Methods in Computational Physics: Volume 3 Fundamental Methods in Hydrodynamics*, Academic Press Inc., 1964.

[32] Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity*, 4th Ed., Dover Publications, New York, 1944.

[33] Eringen, A.C., "Propagations of Elastic Waves Generated by Dynamic Loads on a Circular Cavity," *J. Appl. Mech.*, Vol. 28: 218-222, 1961; see also, Erratum, *J. Appl.*, Vol. 30, p. 149, 1963.

[34] Hart, J.F., et al., Computer Approximations, New York: Wiley, 1968.

Appendix A

Displacement in terms of A_{nk} **and** B_{nk}

Aside from $\exp(-i\omega t)$, since

$$\Phi(r,\theta)=\sum_{n=0}^{\infty}A_nH_n^{(1)}(\alpha r)e^{in\theta}$$

$$\Psi(r,\theta)=\sum_{n=0}^{\infty}B_nH_n^{(1)}(\beta r)e^{in\theta},$$

we have:

$$\begin{aligned}\frac{\partial\phi}{\partial r}&=\sum_{n=0}^{\infty}A_n\frac{\partial}{\partial r}H_n^{(1)}(\alpha r)e^{in\theta}\\&=\sum_{n=0}^{\infty}A_{nk}\frac{1}{r}[-nH_n^{(1)}(\alpha r)+\alpha rH_{n-1}^{(1)}(\alpha r)]e^{in\theta}\end{aligned}$$

$$\frac{\partial\phi}{\partial\theta}=\sum_{n=0}^{\infty}iA_{nk}nH_n^{(1)}(\alpha r)e^{in\theta}$$

$$\begin{aligned}\frac{\partial\psi}{\partial r}&=\sum_{n=0}^{\infty}B_{nk}\frac{\partial}{\partial r}H_n^{(1)}(\beta r)e^{in\theta}\\&=\sum_{n=0}^{\infty}B_{nk}\frac{1}{r}[-nH_n^{(1)}(\beta r)+\beta rH_{n-1}^{(1)}(\beta r)]e^{in\theta}\end{aligned}$$

$$\frac{\partial\psi}{\partial\theta}=\sum_{n=0}^{\infty}iB_{nk}nH_n^{(1)}(\beta r)e^{in\theta}.$$

Then we have:

$$\begin{aligned}u_r&=\frac{\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\\&=\frac{1}{r}\sum_{n=0}^{\infty}\{A_{nk}[-nH_n^{(1)}(\alpha r)+\alpha rH_{n-1}^{(1)}(\alpha r)]+iB_{nk}nH_n^{(1)}(\beta r)\}e^{in\theta}\\&=\frac{1}{r}\sum_{n=0}^{\infty}[A_{nk}U_1^{(3)}(\alpha r)+iB_{nk}U_2^{(3)}(\beta r)]e^{in\theta}\end{aligned}$$

$$\begin{aligned}u_\theta&=\frac{1}{r}\frac{\partial\phi}{\partial\theta}-\frac{\partial\psi}{\partial r}\\&=\frac{1}{r}\sum_{n=0}^{\infty}\{iA_{nk}nH_n^{(1)}(\alpha r)+B_{nk}[nH_n^{(1)}(\beta r)-\beta rH_{n-1}^{(1)}(\beta r)]\}e^{in\theta}\\&=\frac{1}{r}\sum_{n=0}^{\infty}[iA_{nk}V_1^{(3)}(\alpha r)+B_{nk}V_2^{(3)}(\beta r)]e^{in\theta}.\end{aligned}$$

Appendix B

Stresses in terms of A_{nk} and B_{nk}

Plane strain stress-strain relationship is:

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y)$$

$$\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y)$$

$$\sigma_z = \lambda(\varepsilon_x + \varepsilon_y) = \nu(\sigma_x + \sigma_y)$$

$$\sigma_{xy} = G\gamma_{xy}$$

$$\sigma_{xz} = 0$$

$$\sigma_{yz} = 0$$

where shear modulus G and the quantity λ are the Lame constants.

In cylindrical coordinates,

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\varepsilon_z = 0$$

$$\varepsilon_{r\theta} = \frac{1}{2}(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r})$$

$$\varepsilon_{\theta z} = 0$$

$$\varepsilon_{rz} = 0.$$

The stresses in terms of displacements u_r and u_θ are:

$$\sigma_r = (2G+\lambda)\frac{\partial u_r}{\partial r} + \lambda(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r})$$

$$\sigma_\theta = (2G+\lambda)(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}) + \lambda\frac{\partial u_r}{\partial r}$$

$$\sigma_z = \lambda(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r})$$

$$\sigma_{r\theta} = G(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r})$$

$$\sigma_{rz} = 0$$

$$\sigma_{\theta z} = 0.$$

Using the above equations, we can obtain the stresses in the cylindrical coordinates in terms of the unknown parameters A_{nk} and B_{nk}. Aside from $\exp(-i\omega t)$, since:

$$\frac{\partial u_r}{\partial r} = \frac{1}{r^2}\sum_{n=0}^{\infty}\{A_{nk}[-\alpha r H_{n-1}^{(1)}(\alpha r) + (n^2 + n - \alpha^2 r^2)H_n^{(1)}(\alpha r)]$$
$$+iB_{nk}[n\beta r H_{n-1}^{(1)}(\beta r) - (n^2+n)H_n^{(1)}(\beta r)]\}e^{in\theta}$$

$$\frac{\partial u_\theta}{\partial r} = \frac{1}{r^2}\sum_{n=0}^{\infty}\{iA_{nk}[n\alpha r H_{n-1}^{(1)}(\alpha r) - (n^2 + n)H_n^{(1)}(\alpha r)]$$
$$+B_{nk}[\beta r H_{n-1}^{(1)}(\beta r) - (n^2+n-\beta^2 r^2)H_n^{(1)}(\beta r)]\}e^{in\theta}$$

$$\frac{\partial u_r}{\partial \theta} = \frac{1}{r^2}\sum_{n=0}^{\infty}\{iA_{nk}[n\alpha r H_{n-1}^{(1)}(\alpha r) - n^2 H_n^{(1)}(\alpha r)] - B_{nk}n^2 H_n^{(1)}(\beta r)]\}e^{in\theta}$$

$$\frac{\partial u_\theta}{\partial \theta} = \frac{1}{r^2}\sum_{n=0}^{\infty}\{-A_{nk}n^2 H_n^{(1)}(\alpha r) + iB_{nk}[-n\beta r H_{n-1}^{(1)}(\beta r) + n^2 H_n^{(1)}(\beta r)]\}e^{in\theta}$$
,

the final stresses become:

$$\sigma_r = \frac{2\mu}{r^2}\sum_{n=0}^{N}\sum_{k=1}^{L}\{A_{nk}[(n^2+n-\frac{1}{2}\beta^2 r^2)H_n^{(1)}(\alpha r) - \alpha r H_{n-1}^{(1)}(\alpha r)]$$
$$+iB_{nk}[-(n^2+n)H_n^{(1)}(\beta r) + n\beta r H_{n-1}^{(1)}(\beta r)]\}e^{in\theta}$$

$$\sigma_\theta = \frac{2\mu}{r^2}\sum_{n=0}^{N}\sum_{k=1}^{L}\{A_{nk}[-(n^2+n-\frac{1}{2}\beta^2 r^2 - \alpha^2 r^2)H_n^{(1)}(\alpha r) + \alpha r H_{n-1}^{(1)}(\alpha r)]$$
$$+iB_{nk}[(n^2+n)H_n^{(1)}(\beta r) - n\beta r H_{n-1}^{(1)}(\beta r)]\}e^{in\theta}$$

$$\sigma_{r\theta} = \frac{2\mu}{r^2}\sum_{n=0}^{N}\sum_{k=1}^{L}\{[iA_{nk}[-(n^2+n)H_n^{(1)}(\alpha r) + n\alpha r H_{n-1}^{(1)}(\alpha r)]$$
$$+B_{nk}[-(n^2+n-\frac{1}{2}\beta^2 r^2)H_n^{(1)}(\beta r) + \beta r H_{n-1}^{(1)}(\beta r)]\}e^{in\theta}$$

$$\sigma_z = \frac{2\mu}{r^2}\sum_{n=0}^{N}\sum_{k=1}^{L}A_{nk}(\alpha^2 r^2 - \frac{1}{2}\beta^2 r^2)H_n^{(1)}(\alpha r)e^{in\theta}$$

$$\sigma_{rz} = 0$$

$$\sigma_{\theta z} = 0.$$

Appendix C

Matrix formulation for Least-Squares procedures

Define:

$$V_{r_j}(r_j, \theta_j, A_{ik}, B_{ik}) - \overline{V}_{r_j} = h_j$$

$$V_{\theta_j}(r_j, \theta_j, A_{ik}, B_{ik}) - \overline{V}_{\theta_j} = k_j.$$

(I) $\dfrac{\partial F}{\partial A_{ik}} = 0$ case:

$$\sum_{j=1}^{m}\{\frac{\partial(h_j^2)}{\partial A_{ik}} + \frac{\partial(k_j^2)}{\partial A_{ik}}\} = 0$$

$$\sum_{j=1}^{m}(h_j \frac{\partial h_j}{\partial A_{ik}} + k_j \frac{\partial k_j}{\partial A_{ik}}) = 0$$

(II) $\dfrac{\partial F}{\partial B_{ik}} = 0$ case:

$$\sum_{j=1}^{m}\{\frac{\partial(h_j^2)}{\partial B_{ik}} + \frac{\partial(k_j^2)}{\partial B_{ik}}\} = 0$$

$$\sum_{j=1}^{m}(h_j \frac{\partial h_j}{\partial B_{ik}} + k_j \frac{\partial k_j}{\partial B_{ik}}) = 0$$

(1) For A_{ik}

Define:

$$\frac{\partial h_j}{\partial A_{ik}} = p_{j,ik}(r_j, \theta_j)$$

$$\frac{\partial k_j}{\partial A_{ik}} = q_{j,ik}(r_j, \theta_j).$$

Then,

$$[h_1 \quad h_2 \quad h_3 \quad \ldots \quad h_m]\mathbf{P} + [k_1 \quad k_2 \quad k_3 \quad \ldots \quad k_m]\mathbf{Q} = [0 \quad 0 \quad 0 \quad \ldots \quad 0]$$

where,

$$\mathbf{P}=\begin{bmatrix} p_{1,01} & \cdots & p_{1,0l} & p_{1,11} & \cdots & p_{1,1l} & \cdots & p_{1,n1} & \cdots & p_{1,nl} \\ p_{2,01} & \cdots & p_{2,0l} & p_{2,11} & \cdots & p_{2,1l} & \cdots & p_{2,n1} & \cdots & p_{2,nl} \\ \\ \\ p_{m,01} & \cdots & p_{m,0l} & p_{m,11} & \cdots & p_{m,1l} & \cdots & p_{m,n1} & \cdots & p_{m,nl} \end{bmatrix}$$

$$\mathbf{Q}=\begin{bmatrix} q_{1,01} & \cdots & q_{1,0l} & q_{1,11} & \cdots & q_{1,1l} & \cdots & q_{1,n1} & \cdots & q_{1,nl} \\ q_{2,01} & \cdots & q_{2,0l} & q_{2,11} & \cdots & q_{2,1l} & \cdots & q_{2,n1} & \cdots & q_{2,nl} \\ \\ \\ q_{m,01} & \cdots & q_{m,0l} & q_{m,11} & \cdots & q_{m,1l} & \cdots & q_{m,n1} & \cdots & q_{m,nl} \end{bmatrix}.$$

Also define:

$$\begin{bmatrix} h_1 & h_2 & h_3 & \ldots & h_m \end{bmatrix}=\mathbf{H}$$

$$\begin{bmatrix} k_1 & k_2 & k_3 & \ldots & k_m \end{bmatrix}=\mathbf{K}.$$

Then we have:

$$\mathbf{HP}+\mathbf{KQ}=\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Transposing gives:

$$(\mathbf{HP}+\mathbf{KQ})^T=\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \end{bmatrix}^T$$

$$(\mathbf{HP})^T+(\mathbf{KQ})^T=\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \end{bmatrix}^T.$$

We now have:

$$\mathbf{P}^T\mathbf{H}^T+\mathbf{Q}^T\mathbf{K}^T=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \end{bmatrix}$$

$$\mathbf{P}^T\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \\ h_m \end{bmatrix}+\mathbf{Q}^T\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \\ k_m \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \end{bmatrix}.$$

Since we have:

$$\begin{bmatrix} V_{r_1} \\ V_{r_2} \\ V_{r_3} \\ \vdots \\ V_{r_m} \end{bmatrix} - \begin{bmatrix} \overline{V}_{r_1} \\ \overline{V}_{r_2} \\ \overline{V}_{r_3} \\ \vdots \\ \overline{V}_{r_m} \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_m \end{bmatrix} \text{ and } \begin{bmatrix} V_{\theta_1} \\ V_{\theta_2} \\ V_{\theta_3} \\ \vdots \\ V_{\theta_m} \end{bmatrix} - \begin{bmatrix} \overline{V}_{\theta_1} \\ \overline{V}_{\theta_2} \\ \overline{V}_{\theta_3} \\ \vdots \\ \overline{V}_{\theta_m} \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_m \end{bmatrix},$$

we finally have:

$$\mathbf{P}^T \begin{bmatrix} V_{r_1} \\ V_{r_2} \\ V_{r_3} \\ \vdots \\ V_{r_m} \end{bmatrix} + \mathbf{Q}^T \begin{bmatrix} V_{\theta_1} \\ V_{\theta_2} \\ V_{\theta_3} \\ \vdots \\ V_{\theta_m} \end{bmatrix} = \mathbf{P}^T \begin{bmatrix} \overline{V}_{r_1} \\ \overline{V}_{r_2} \\ \overline{V}_{r_3} \\ \vdots \\ \overline{V}_{r_m} \end{bmatrix} + \mathbf{Q}^T \begin{bmatrix} \overline{V}_{\theta_1} \\ \overline{V}_{\theta_2} \\ \overline{V}_{\theta_3} \\ \vdots \\ \overline{V}_{\theta_m} \end{bmatrix}.$$

(2) For B_{ik}

Define:

$$\frac{\partial h_j}{\partial B_{ik}} = c_{j,ik}(r_j, \theta_j)$$

$$\frac{\partial k_j}{\partial B_{ik}} = d_{j,ik}(r_j, \theta_j).$$

Then,

$$[h_1 \quad h_2 \quad h_3 \quad \ldots \quad h_m]\mathbf{C} + [k_1 \quad k_2 \quad k_3 \quad \ldots \quad k_m]\mathbf{D} = [0 \quad 0 \quad 0 \quad \ldots \quad 0].$$

Matrix $\mathbf{C}$ and $\mathbf{D}$ are same patern as $\mathbf{P}$ and $\mathbf{Q}$. Then,

$$\mathbf{HC} + \mathbf{KD} = [0 \quad 0 \quad 0 \quad \ldots \quad 0].$$

Transposing gives:

$$(\mathbf{HC} + \mathbf{KD})^T = [0 \quad 0 \quad 0 \quad \ldots \quad 0]^T$$

$$(\mathbf{HC})^T + (\mathbf{KD})^T = [0 \quad 0 \quad 0 \quad \ldots \quad 0]^T.$$

We now have:

$$\mathbf{C}^T\mathbf{H}^T + \mathbf{D}^T\mathbf{K}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \end{bmatrix}$$

$$\mathbf{C}^T \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \\ h_m \end{bmatrix} + \mathbf{D}^T \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \\ k_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \end{bmatrix}.$$

We finally have:

$$\mathbf{C}^T \begin{bmatrix} V_{r_1} \\ V_{r_2} \\ V_{r_3} \\ \vdots \\ V_{r_m} \end{bmatrix} + \mathbf{D}^T \begin{bmatrix} V_{\theta_1} \\ V_{\theta_2} \\ V_{\theta_3} \\ \vdots \\ V_{\theta_m} \end{bmatrix} = \mathbf{C}^T \begin{bmatrix} \overline{V}_{r_1} \\ \overline{V}_{r_2} \\ \overline{V}_{r_3} \\ \vdots \\ \overline{V}_{r_m} \end{bmatrix} + \mathbf{D}^T \begin{bmatrix} \overline{V}_{\theta_1} \\ \overline{V}_{\theta_2} \\ \overline{V}_{\theta_3} \\ \vdots \\ \overline{V}_{\theta_m} \end{bmatrix}.$$

If we define the column vector of unknown constants A_{ik} and B_{ik} as $\mathbf{A}$ and $\mathbf{B}$, respectively, i.e.,

$$\begin{bmatrix} A_{01} \\ A_{02} \\ \\ A_{0l} \\ \vdots \\ A_{n1} \\ A_{n2} \\ \\ A_{nl} \end{bmatrix} = \mathbf{A} \quad \text{and} \quad \begin{bmatrix} B_{01} \\ B_{02} \\ \\ B_{0l} \\ \vdots \\ B_{n1} \\ B_{n2} \\ \\ B_{nl} \end{bmatrix} = \mathbf{B},$$

we have the velocity matrices expressed in terms of $\mathbf{A}$ and $\mathbf{B}$ as shown below:

$$\begin{bmatrix} V_{r_1} \\ V_{r_2} \\ V_{r_3} \\ \vdots \\ V_{r_m} \end{bmatrix} = \mathbf{P} \begin{bmatrix} A_{01} \\ A_{02} \\ \\ A_{0l} \\ \vdots \\ A_{n1} \\ A_{n2} \\ \\ A_{nl} \end{bmatrix} + \mathbf{C} \begin{bmatrix} B_{01} \\ B_{02} \\ \\ B_{0l} \\ \vdots \\ B_{n1} \\ B_{n2} \\ \\ B_{nl} \end{bmatrix} = \mathbf{PA} + \mathbf{CB}$$

and

$$\begin{bmatrix} V_{\theta_1} \\ V_{\theta_2} \\ V_{\theta_3} \\ \vdots \\ V_{\theta_m} \end{bmatrix} = \mathbf{Q} \begin{bmatrix} A_{01} \\ A_{02} \\ \\ A_{0l} \\ \vdots \\ A_{n1} \\ A_{n2} \\ \\ A_{nl} \end{bmatrix} + \mathbf{D} \begin{bmatrix} B_{01} \\ B_{02} \\ \\ B_{0l} \\ \vdots \\ B_{n1} \\ B_{n2} \\ \\ B_{nl} \end{bmatrix} = \mathbf{QA} + \mathbf{DB}$$

Now let's also define the column vector of the specified velocities at each node point as:

$$\begin{bmatrix} \overline{V}_{r_1} \\ \overline{V}_{r_2} \\ \overline{V}_{r_3} \\ \vdots \\ \overline{V}_{r_m} \end{bmatrix} = \mathbf{S} \text{ and } \begin{bmatrix} \overline{V}_{\theta_1} \\ \overline{V}_{\theta_2} \\ \overline{V}_{\theta_3} \\ \vdots \\ \overline{V}_{\theta_m} \end{bmatrix} = \mathbf{T}.$$

Then two final matrix forms are:

(1) For A_{ik} :

$$\mathbf{P}^T(\mathbf{PA} + \mathbf{CB}) + \mathbf{Q}^T(\mathbf{QA} + \mathbf{DB}) = \mathbf{P}^T\mathbf{S} + \mathbf{Q}^T\mathbf{T}$$

$$(\mathbf{P}^T\mathbf{P} + \mathbf{Q}^T\mathbf{Q})\mathbf{A} + (\mathbf{P}^T\mathbf{C} + \mathbf{Q}^T\mathbf{D})\mathbf{B} = \mathbf{P}^T\mathbf{S} + \mathbf{Q}^T\mathbf{T}$$

(2) For B_{ik} :

$$\mathbf{C}^T(\mathbf{PA} + \mathbf{CB}) + \mathbf{D}^T(\mathbf{QA} + \mathbf{DB}) = \mathbf{C}^T\mathbf{S} + \mathbf{D}^T\mathbf{T}$$

$$(\mathbf{C}^T\mathbf{P}+\mathbf{D}^T\mathbf{Q})\mathbf{A}+(\mathbf{C}^T\mathbf{C}+\mathbf{D}^T\mathbf{D})\mathbf{B}=\mathbf{C}^T\mathbf{S}+\mathbf{D}^T\mathbf{T}$$

For both equations, the matrix form becomes:

$$\begin{bmatrix}\mathbf{P}^T\mathbf{P}+\mathbf{Q}^T\mathbf{Q} & \mathbf{P}^T\mathbf{C}+\mathbf{Q}^T\mathbf{D}\\ \mathbf{C}^T\mathbf{P}+\mathbf{D}^T\mathbf{Q} & \mathbf{C}^T\mathbf{C}+\mathbf{D}^T\mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}=\begin{bmatrix}\mathbf{P}^T & \mathbf{Q}^T\\ \mathbf{C}^T & \mathbf{D}^T\end{bmatrix}\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix}.$$

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